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A NOTE ON KOOP'S PROCEDURE TO OBTAIN THE BIAS OF THE RATIO ESTIMATE

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SUMMARY. Koop (1951) gave a method to find the bias of the ratio estimate. The feature of his procedure to obtain the bias is in the use of unselected sample. But his treatment of an infinite series is wrong and conclusion is not valid. This paper presents a proof of this.

1. Introduction

First, we have to assure of the nature of an infinite series which is convergent but not absolutely (conditionally convergent). According to Riemann's theorem, if we change the adding order over infinite number of terms, the series can have another sum or possibly divergent. For example, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ is convergent and famous for its behavior by changing the order of addition.

2. Remarks on Koop's procedure

For the sake of convenience, we shall use Koop's notation and discuss on the lines given by him.

Koop carried out $E\left(\frac{\bar{x}}{\bar{y}}\right)$ as follows :

$$E\left(\frac{\bar{x}}{\bar{y}}\right)=\frac{\mu_{10}}{\mu_{01}}\left[1+\sum_{r=1}^{\infty}f^r\left\{E\left(\frac{\bar{y}'^r}{\mu_{01}^r}\right)-E\left(\frac{\bar{x}'\bar{y}'^{r-1}}{\mu_{10}\mu_{01}^{r-1}}\right)\right\}\right]. \qquad \ldots \quad (1)$$

This series is valid for any $\bar{x}'$ and $\bar{y}'$. The two expectations in { } of (1) are developed into finite series of binomial type, i.e.,

$$E\left(\frac{\bar{y}'^r}{\mu_{01}^r}\right)=E\left(1+\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^r=\sum_{i=0}^{r}\binom{r}{i}E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i \qquad \ldots \quad (2)$$

$$E\left(\frac{\bar{x}'\bar{y}'^{r-1}}{\mu_{10}\mu_{01}^{r-1}}\right)=E\left(1+\frac{\bar{x}'-\mu_{10}}{\mu_{10}}\right)\left(1+\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^{r-1}$$

$$=\sum_{i=0}^{r-1}\left\{\binom{r-1}{i}E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i+\binom{r-1}{i}E\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^i}{\mu_{10}\mu_{01}^{\,i}}\right\}. \qquad \ldots \quad (3)$$

Substituting from (2) and (3) in (1), we have

$$E\left(\frac{\bar{x}}{\bar{y}}\right)=\frac{\mu_{10}}{\mu_{01}}\left[1+\sum_{r=1}^{\infty}\sum_{i=1}^{r}\left\{f^r\binom{r-1}{i-1}E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i\right.\right.$$

$$\left.\left.-f^r\binom{r-1}{i-1}E\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^{i-1}}{\mu_{10}\mu_{01}^{i-1}}\right\}\right]. \qquad \ldots \quad (4)$$

For simplicity, let us denote by a_{ri} the quantity in { } of (4), then (4) is expressed as

$$E\left(\frac{\bar{x}}{\bar{y}}\right) = \frac{\mu_{10}}{\mu_{01}}\left[1+\sum_{r=1}^{\infty}\sum_{i=1}^{r} a_{ri}\right]. \qquad \dots \quad (5)$$

This is still valid under the normal order of addition. The adding pattern of (5) is visibly shown in Fig. 1 where the arrows mean the order of addition.

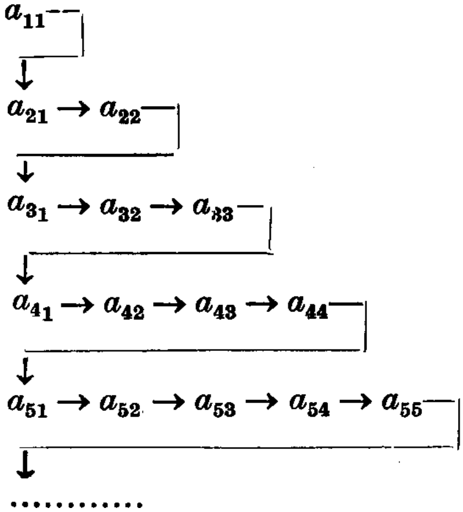

Fig. 1

According to Koop's idea, we shall try to change the order of addition as shown in Fig. 2. The double summation of a_{ri} in (5) will then be changed to $\sum_{i=1}^{\infty}\sum_{r=i}^{\infty}$. Here, our attention has to be paid whether $\sum_{r=1}^{\infty}\sum_{i=1}^{r} a_{ri}$ is identical to $\sum_{i=1}^{\infty}\sum_{r=i}^{\infty} a_{ri}$. As seen below, these two are not quite identical.

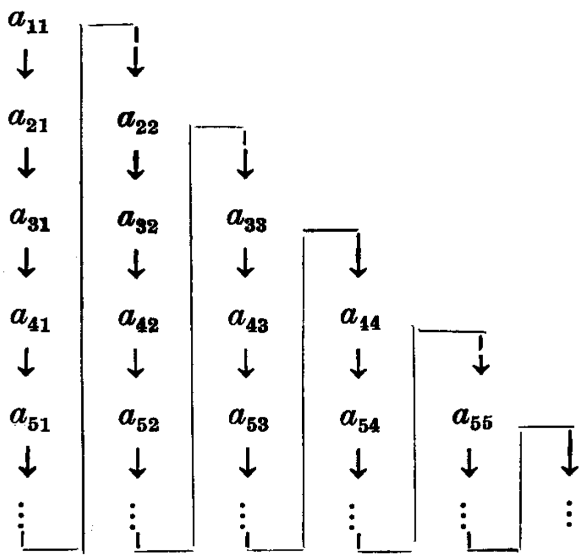

Fig. 2

Resuming (4), let us assume that the rearrangement of terms by Koop's way is admissible. Then we can promote the process as follows.

$$E\left(\frac{\bar{x}}{\bar{y}}\right) = \frac{\mu_{10}}{\mu_{01}}\Big[1+\sum_{i=1}^{\infty}\sum_{r=i}^{\infty}\Big\{f^r\binom{r-1}{i-1}E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i$$

$$-f^r\binom{r-1}{i-1}E\,\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^{i-1}}{\mu_{10}\mu_{01}{}^{i-1}}\Big\}\Big]$$

$$=\frac{\mu_{10}}{\mu_{01}}\Big[1+\sum_{i=1}^{\infty}\Big\{\Big(\sum_{r=i}^{\infty}f^r\binom{r-1}{i-1}\Big)E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i\Big\}$$

$$-\sum_{i=1}^{\infty}\Big\{\Big(\sum_{r=i}^{\infty}f^r\binom{r-1}{i-1}\Big)E\,\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^{i-1}}{\mu_{10}\mu_{01}^{i-1}}\Big\}\Big] \qquad \ldots \quad (6)$$

Remembering the negative binomial series

$$\sum_{r=i}^{\infty}f^r\binom{r-1}{i-1}=f^i+if^{i+1}+\frac{i(i+1)}{2\,!}f^{i+2}+\frac{i(i+1)(i+2)}{3\,!}f^{i+3}+\ldots$$

$$=f^i(1-f)^{-i} \quad (\because\; 0<f<1). \qquad \ldots \quad (7)$$

Hence, (6) becomes

$$E\left(\frac{\bar{x}}{\bar{y}}\right) = \frac{\mu_{10}}{\mu_{01}}\Big[1+\sum_{i=1}^{\infty}\Big\{\left(\frac{f}{1-f}\right)^i E\left(\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i\Big\}$$

$$-\sum_{i=1}^{\infty}\Big\{\left(\frac{f}{1-f}\right)^i E\,\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^{i-1}}{\mu_{10}\mu_{01}^{i-1}}\Big\}\Big]$$

$$=\frac{\mu_{10}}{\mu_{01}}\Big[1+\sum_{i=1}^{\infty}E\left(\frac{f}{1-f}\,\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right)^i$$

$$-\sum_{i=1}^{\infty}E\left(\frac{f}{1-f}\right)^i\frac{(\bar{x}'-\mu_{10})(\bar{y}'-\mu_{01})^{i-1}}{\mu_{10}\mu_{01}^{i-1}}\Big] \qquad \ldots \quad (8)$$

However, for the validity of (8), the inequality

$$\left|\frac{f}{1-f}\,\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right|<1 \qquad \ldots \quad (9)$$

must be satisfied. Since (4) does not require (9), we find that (4) and (8) are no longer identical. This is inconsistent with the assumption that the rearrangement of terms by Koop's way is admissible.

Further, since $N\mu_{01}=(N-n)\bar{y}'+n\bar{y}$ and $f=\dfrac{N-n}{N}$, we have

$$\left|\frac{f}{1-f}\,\frac{\bar{y}'-\mu_{01}}{\mu_{01}}\right|=\left|\frac{\bar{y}-\mu_{01}}{\mu_{01}}\right|<1. \qquad \ldots \quad (10)$$

Under this condition, (8) becomes quite identical to Koop's final formula and also to what is obtained from the conventional procedure by expanding $\left(1+\frac{\bar{y}-\mu_{01}}{\mu_{01}}\right)^{-1}$. If (10) is not satisfied, then (8) is divergent. Hence, we finally know that Koop's conclusion that the approximate expressions for the bias obtained by assuming $\left|\frac{\bar{y}-\mu_{01}}{\mu_{01}}\right| < 1$ and by not assuming this restriction are identical is not valid. Hence, there is no paradox in this situation.

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Reference

Koop, J. C. (1951): A note on the bias of the ratio estimate. *Bull. Int. Stat. Inst.*. 33, Pt. II, 141-146.

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