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A METHOD OF DERIVING VALID APPROXIMATE EXPRESSIONS FOR BIAS IN RATIO ESTIMATION

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Abstract: The expectation of a ratio of two statistics is usually obtained by the method in which the denominator is expanded into a binomial series around its mean. However, this method can less generally be employed because the expansion of the denominator is not always valid. This paper presents a device in which the expectation of a ratio of two statistics can more generally be obtained. The device is to adopt a positive value as 'catalyzer'. It can be applied not only to the ratio of estimates based on samples but also to ratio of any two statistics, if the denominator is positive and finite.

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1. Introduction

It is a quite basic matter to obtain the bias of a ratio of two statistics. Koop (1951) presented a formula of bias in the ratio of two sample means in detail and gave a conclusion that the expressions for the bias obtained by assuming restriction in the denominator's range of variation and by not assuming this restriction are identical. However, Funatsu (1980) objected to Koop's theory. The theory below is to be an alternative theory to Koop's. By the theory below, conventional theories such as Murthy's (1962) regarding to the bias in non-linear function of estimators will be much generalized.

Suppose there are two statistics $\hat{Y}_1$ and $\hat{Y}_2$.

Let $\hat{Y}_1$ be an unbiased estimator of Y_1 that is not equal to zero, i.e.

$$E(\hat{Y}_1) = Y_1 \neq 0$$

and $\hat{Y}_2$ be an unbiased estimator of Y_2 such that

$$\Pr(L_1 < \hat{Y}_2 < L_2) = 1 \tag{1}$$

where L_1 and L_2 are positive values so fixed as to make their difference $L_2 - L_1$ small. If $\hat{Y}_2$ has its lower bound $Y_{2(1)}$, and the upper bound $Y_{2(2)}$, then L_1 and L_2 can

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be expressed as

$$L_1 = Y_{2(1)} - \varepsilon, \qquad L_2 = Y_{2(2)} + \varepsilon \tag{2}$$

where ε is a small positive value (see Fig. 1).

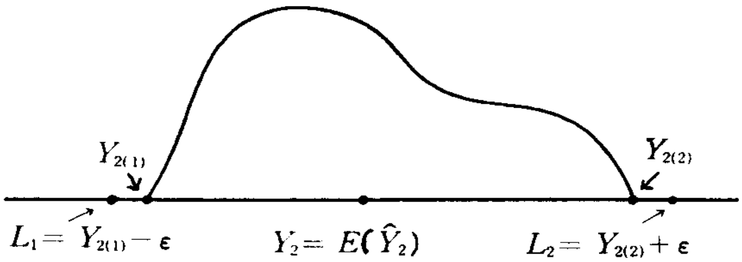

Fig. 1. Probability distribution of $\hat{Y}_2$.

From (1), we see

$$L_1 < Y_2 < L_2 \tag{3}$$

and we can find another fixed positive value k which satisfies

$$L_2 < 2kY_2. \tag{4}$$

From (3) and (4), we have

$$k > \frac{L_2}{2Y_2} > \frac{1}{2} \tag{5}$$

2. Extended concept of the moment

Consider a variable $\hat{Y}_1$. We already know the two kinds of moments $E(\hat{Y}_1^{\alpha})$ and $E(\hat{Y}_1 - Y_1)^{\alpha}$ where Y_1 is the mean of $\hat{Y}_1$ and α is the degree taking 0, 1, 2, 3, ….

Thus, we generally use zero or the mean as the base (center) of the moment. However, we need not adhere to zero or the mean. By using λ_1, we can define more generally $E(\hat{Y}_1 - \lambda_1)^{\alpha}$. Clearly, this can be expressed by the conventional moments no higher than α-th degree. Similarly, we can define $E(\hat{Y}_1 - \lambda_1)^{\alpha}(\hat{Y}_2 - \lambda_2)^{\beta}$ for the joint distribution of two variables $\hat{Y}_1$ and $\hat{Y}_2$. The series of these extended moments, of course, characterizes their population. And these can be estimated by a sample.

In the discussion below, we shall use the following type of moment defined by

$$E(\hat{Y}_1 - kY_1)^{\alpha}(\hat{Y}_2 - kY_2)^{\beta} = \sigma_{\alpha\beta}(k)$$
$$\alpha = 0, 1, 2, \ldots, \quad \beta = 0, 1, 2, \ldots, \quad k > \frac{L_2}{2Y_2} > \frac{1}{2}.$$

When $k = 1$, the above moment resumes the conventional form, and writing

$$E(\hat{Y}_1 - Y_1)^{\alpha}(\hat{Y}_2 - Y_2)^{\beta} = \sigma_{\alpha\beta},$$

$\sigma_{\alpha\beta}(k)$ is expressed by using the conventional moments and product-moments. For example, when $\alpha = 0, 1, 2$ and $\beta = 0, 1, 2, 3, 4$, we have (up to fourth degree)

$$\left.\begin{aligned}
&\sigma_{00}(k)=1, \qquad \sigma_{10}(k)=-(k-1)Y_1, \qquad \sigma_{01}(k)=-(k-1)Y_2,\\
&\sigma_{20}(k)=\sigma_{20}+(k-1)^2Y_1^2,\\
&\sigma_{11}(k)=\sigma_{11}+(k-1)^2Y_1Y_2,\\
&\sigma_{02}(k)=\sigma_{02}+(k-1)^2Y_2^2,\\
&\sigma_{21}(k)=\sigma_{21}-2(k-1)Y_1\sigma_{11}-(k-1)Y_2\sigma_{20}-(k-1)^3Y_1^2Y_2,\\
&\sigma_{12}(k)=\sigma_{12}-2(k-1)Y_2\sigma_{11}-(k-1)Y_1\sigma_{02}-(k-1)^3Y_1Y_2^2,\\
&\sigma_{03}(k)=\sigma_{03}-3(k-1)Y_2\sigma_{02}-(k-1)^3Y_2^3,\\
&\sigma_{22}(k)=\sigma_{22}-2(k-1)Y_1\sigma_{12}-2(k-1)Y_2\sigma_{21}+(k-1)^2Y_1^2\sigma_{02}\\
&\qquad\quad+(k-1)^2Y_2^2\sigma_{20}+4(k-1)^2Y_1Y_2\sigma_{11}+(k-1)^4Y_1^2Y_2^2,\\
&\sigma_{13}(k)=\sigma_{13}-3(k-1)Y_2\sigma_{12}-(k-1)Y_1\sigma_{03}+3(k-1)^2Y_2^2\sigma_{11}\\
&\qquad\quad+3(k-1)^2Y_1Y_2\sigma_{02}+(k-1)^4Y_1Y_2^3,\\
&\sigma_{04}(k)=\sigma_{04}-4(k-1)Y_2\sigma_{03}+6(k-1)^2Y_2^2\sigma_{02}+(k-1)^4Y_2^4.
\end{aligned}\right\} \tag{6}$$

3. The ratio and its expectation

Let us denote by $R = Y_1/Y_2$ the ratio of Y_1 to Y_2, and $\hat{R} = \hat{Y}_1/\hat{Y}_2$ the corresponding ratio of the estimators used for the estimation of R. In order to obtain the expectation of $\hat{R}$, first we rewrite $\hat{R}$ as follows:

$$\hat{R}=\frac{\hat{Y}_1}{\hat{Y}_2}=\frac{kY_1+\hat{Y}_1-kY_1}{kY_2+\hat{Y}_2-kY_2}=R\left(1+\frac{\hat{Y}_1-kY_1}{kY_1}\right)\left(1+\frac{\hat{Y}_2-kY_2}{kY_2}\right)^{-1}. \tag{7}$$

From (1) and (4)

$$\frac{\hat{Y}_2-kY_2}{kY_2}>\frac{L_1-kY_2}{kY_2}>-1,$$

$$\frac{\hat{Y}_2-kY_2}{kY_2}<\frac{L_2-kY_2}{kY_2}<\frac{2kY_2-kY_2}{kY_2}=1.$$

Therefore

$$-1<\frac{\hat{Y}_2-kY_2}{kY_2}<1. \tag{8}$$

Hence, the expansion of

$$\left(1+\frac{\hat{Y}_2-kY_2}{kY_2}\right)^{-1}$$

becomes valid. Expanding this, we have

$$\left(1+\frac{\hat{Y}_2-kY_2}{kY_2}\right)^{-1}=\sum_{i=0}^{\infty}(-1)^i\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i$$

and

$$\hat{R}=R\left(1+\frac{\hat{Y}_1-kY_1}{kY_1}\right)\sum_{i=0}^{\infty}(-1)^i\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i$$
$$=R\sum_{i=0}^{\infty}(-1)^i\left\{\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i+\left(\frac{\hat{Y}_1-kY_1}{kY_1}\right)\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i\right\}. \tag{9}$$

Taking expectations term by term in (9), we have

$$E(\hat{R})=R\sum_{i=0}^{\infty}(-1)^i\left\{E\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i+E\left(\frac{\hat{Y}_1-kY_1}{kY_1}\right)\left(\frac{\hat{Y}_2-kY_2}{kY_2}\right)^i\right\}$$
$$=R\sum_{i=0}^{\infty}(-1)^i\left\{\frac{\sigma_{0i}(k)}{(kY_2)^i}+\frac{\sigma_{1i}(k)}{kY_1(kY_2)^i}\right\}$$
$$=R\left[1+\frac{1}{k}\sum_{i=1}^{\infty}\left(\frac{-1}{kY_2}\right)^i\left\{\frac{\sigma_{1i}(k)}{Y_1}-\frac{\sigma_{0,i+1}(k)}{Y_2}\right\}\right] \tag{10}$$
$$\text{where } k>\frac{L_2}{2Y_2}>\frac{1}{2}.$$

(7) is not the only expression of $\hat{R}$. We will have various expressions according to the initial form of expansion. For instance, if we employ

$$\hat{R}=\frac{\hat{Y}_1}{\hat{Y}_2}=\frac{Y_1+\hat{Y}_1-\hat{Y}_1}{kY_2+\hat{Y}_2-kY_2}=R\frac{1}{k}\left(1+\frac{\hat{Y}_1-Y_1}{Y_1}\right)\left(1+\frac{\hat{Y}_2-kY_2}{kY_2}\right)^{-1}$$

we then have another expression of $E(\hat{R})$.

4. Approximate expressions for $E(\hat{R})$

As seen in (10), the expectation of $\hat{R}$ is expressed by an infinite series which is convergent and contains k. But k has no effect on the sum of (10), but influences the rapidity of convergence. In this sense, we call k a 'catalyzer'. However, in practice, since we take only the first few terms up to certain degree, k has an effect on their sum.

The approximation in (10) should be done considering the accuracy of approximation. The accuracy can be measured by the relative discrepancy between the approximate value and the true one. From practical point of view, it will be convenient to take the number of accurate figures (after rounding off) in the approximate value as 'accuracy' (see Table 1).

Needless to say, we will not be satisfied with the accuracy of $s=0$.

Since (10) is convergent, if we give s and k, the minimum degree d up to which the

Table 1
Comparison of approximate and true values for different degree of accuracy

Approximate value	185.5	2.954	0.826	0.04361	37768
True value	590.7	3.101	0.834	0.04362	37771
Degree of accuracy s	0	1	2	3	4

addition has to be operated is determined. Thus we know that d is a function of s and k, i.e.

$$d = f(s, k) \tag{11}$$
$$s = 0, 1, 2, 3, \ldots, \quad k > \frac{L_2}{2Y_2} > \frac{1}{2}, \quad d = (0), 2, 3, \ldots.$$

Since (10) is a convergent infinite series, we can certainly find a finite series for approximation under an appropriate choice of s, k and d. Practically, s should be given first at our disposal. Consequently, our interest is to find the interval of k a value in which minimizes d. We call such interval 'optimum interval' of k. Now, let k_0 be a value in the optimum interval and d_0 the corresponding value of d. Thus, we obtain a set of (s_0, k_0, d_0) where s_0 = accuracy, k_0 = optimum catalyzer, and d_0 = degree of approximation. k_0 and d_0 are dependent on the joint probability distribution of $\hat{Y}_1$ and $\hat{Y}_2$.

Using k_0 and d_0, the approximate expression of (10) is given as follows:

$$E(\hat{R}) := R\left[1 + \frac{1}{k_0}\sum_{i=1}^{d_0-1}\left(\frac{-1}{k_0 Y_2}\right)^i \left\{\frac{\sigma_{1i}(k_0)}{Y_1} - \frac{\sigma_{0,i+1}(k_0)}{Y_2}\right\}\right] \tag{12}$$
$$(d_0 \geq 2).$$

The moment $\sigma_{0,i+1}(k_0)$ and the product-moment $\sigma_{1i}(k_0)$ can be expressed in terms of conventional ones by (6) and the like. Thus, we have the second degree of approximation

$$E(\hat{R}) := R\left\{1 + \frac{1}{k_0^2}\left(\frac{\sigma_{02}}{Y_2^2} - \frac{\sigma_{11}}{Y_1 Y_2}\right)\right\} \tag{13}$$

corresponding to $d_0 = 2$, the third degree of approximation

$$E(\hat{R}) := R\left\{1 + \frac{3k_0 - 2}{k_0^3}\left(\frac{\sigma_{02}}{Y_2^2} - \frac{\sigma_{11}}{Y_1 Y_2}\right) - \frac{1}{k_0^3}\left(\frac{\sigma_{03}}{Y_2^3} - \frac{\sigma_{12}}{Y_1 Y_2^2}\right)\right\} \tag{14}$$

with $d_0 = 3$, and the fourth degree of approximation

$$E(\hat{R}) := R\left\{1 + \frac{6k_0^2 - 8k_0 + 3}{k_0^4}\left(\frac{\sigma_{02}}{Y_2^2} - \frac{\sigma_{11}}{Y_1 Y_2}\right) - \frac{4k_0 - 3}{k_0^4}\left(\frac{\sigma_{03}}{Y_2^3} - \frac{\sigma_{12}}{Y_1 Y_2^2}\right) + \frac{1}{k_0^4}\left(\frac{\sigma_{04}}{Y_2^4} - \frac{\sigma_{13}}{Y_1 Y_2^3}\right)\right\} \tag{15}$$

for $d_0=4$.

Thus, we have approximate expressions of $E(\hat{R})$ as given in (13), (14) and (15). Needless to say, (14) is better than (13), and (15) is better than (14) under our definition of accuracy. But, we have here to pay attention to these formulas.

Firstly, suppose (13) is applied to a sampling distribution, and $k_0>1$ in certain size n. In this situation, if the sample size increases, then k_0 will approach unity and it seems as if the bias in $\hat{R}$ got greater. But, actually, the bias decreases, since σ_{02}, σ_{11} and $E(\hat{R})$ vary when n varies, while Y_1, Y_2 and R are constant.

Secondly, as seen in (13) and (14), if $k_0>1$, then

$$\frac{3k_0-2}{k_0^3}>\frac{1}{k_0^2}.$$

This does not mean that (13) is a better approximation than (14). Actually, (14) is still better than (13), adjusted by the third degree of the moment and the product-moment. Similarly, (15) is better than (14). These are illustrated by the numerical study in Section 6.

5. Approximation for k_0

To find the value of k_0, from (5), we shall write in a practical form

$$k_0=\frac{L_2+C_0L_1}{2Y_2} \tag{16}$$

where $C_0>0$. It follows that

$$\frac{\hat{Y}_2-k_0Y_2}{k_0Y_2}=\frac{2\hat{Y}_2-(L_2+C_0L_1)}{L_2+C_0L_1}$$

and from (1)

$$-1<-\frac{L_2-(2-C_0)L_1}{L_2+C_0L_1}<\frac{\hat{Y}_2-k_0Y_2}{k_0Y_2}<\frac{L_2-C_0L_1}{L_2+C_0L_1}<1.$$

For rapid convergence of the series (10), it is desirable that $(\hat{Y}_2-k_0Y_2)/k_0Y_2$ has a small range of variation around zero. Therefore, if we put $C_0=1$ as an intuitive decision, then $(\hat{Y}_2-k_0Y_2)/k_0Y_2$ varies within $\pm(L_2-L_1)/(L_2+L_1)$. If $C_0=1$, the right hand side of (16) becomes $(L_2+L_1)/2Y_2$. The reason why we said in Section 1 that L_1 and L_2 are so fixed as to make their difference small lies here. Hence, $(L_2+L_1)/2Y_2$ becomes one of our reasonable approximation for k_0. However, $(L_2+L_1)/2Y_2$ may not be optimum. The optimum values will be found by further study.

When $\hat{Y}_2$ has the lower bound $Y_{2(1)}$ and the upper bound $Y_{2(2)}$, from (2), we have

$$k_0:=\frac{Y_{2(2)}+\varepsilon+Y_{2(1)}-\varepsilon}{2Y_2}=\frac{Y_{2(1)}+Y_{2(2)}}{2Y_2}.$$

Further, when the probability distribution of $\hat{Y}_2$ is symmetric, then

$$\tfrac{1}{2}(Y_{2(1)}+Y_{2(2)})=Y_2$$

and $k_0:=1$. When $k_0=1$, it follows that $|\hat{Y}_2-Y_2|/Y_2<1$. Consequently, (10) resumes the conventional expression. Particularly, if $\hat{Y}_2$ is the mean of a simple random sample under the condition that the sample size is large *and* the sampling fraction is small or zero (when independently sampled), the probability distribution of $\hat{Y}_2$ becomes approximately normal and symmetric by the effect of central limit. It is also known that if the sampling fraction tends to $\frac{1}{2}$, then the probability distribution of the sample mean becomes approximately symmetric irrespective of sample size. Therefore, we can say that it will be reasonable to put $k_0=1$ when the sample size *or* the sampling fraction is fairly large. This coincides with our common sense.

In connection with the choice of k_0, it looks also reasonable to choose k that minimizes $E\{(\hat{Y}_2-kY_2)/kY_2\}^2$ or $E|(\hat{Y}_2-kY_2)/kY_2|$. But this idea is not always valid, because k can become less than $L_2/2Y_2$.

The theory developed above is applicable to the ratio of two sample means. For simplicity, we shall consider this ratio when a simple random sample of size n from a population of size N is selected. Since k_0 and d_0 are dependent on the joint probability distribution of the two sample means, k_0 and d_0 are also dependent on n. Now, we shall write here $k_0(n)$ and $d_0(n)$ instead of k_0 and d_0, respectively. It is obvious that $d_0(n)$ decreases as n increases. According to the above discussion, $k_0(n)$ approaches to unity as n increases. Moreover, when n is fairly large, $k_0(n)$ will change little for large changes of n. As an example, if $d_0(n)=2$ under given accuracy s_0, we have the second degree of approximation as follows:

$$E(\hat{R}):=R\left[1+\frac{1}{\{k_0(n)\}^2}\frac{N-n}{N-1}\frac{1}{n}\left(\frac{\sigma_{02}}{Y_2^2}-\frac{\sigma_{11}}{Y_1Y_2}\right)\right]$$

where

$$\hat{R}=\frac{\bar{Y}_1}{\bar{Y}_2},\qquad R=\frac{E(\bar{Y}_1)}{E(\bar{Y}_2)},\qquad E(\bar{Y}_1)=Y_1,\qquad E(\bar{Y}_2)=Y_2,$$

$\bar{Y}_1$ = sample mean of $\hat{Y}_1$ with size n,

$\bar{Y}_2$ = sample mean of $\hat{Y}_2$ with size n,

$$\sigma_{02}=E(\hat{Y}_2-Y_2)^2,\qquad \sigma_{11}=E(\hat{Y}_1-Y_1)(\hat{Y}_2-Y_2).$$

According to empirical study, when n becomes large, although $k_0(n)$ tends to unity, the coefficient $(N-n)/\{k_0(n)\}^2(N-1)n$ still decreases and the bias in $\hat{R}$ becomes small as a whole, since $k_0(n)$ changes so little.

6. A numerical study

For the practical use of (10), we shall consider data on household size and monthly expenditure for a population of 128 households given by Murthy (1972). In the study below, we pay attention to

(1) lower bound of k,
(2) approximate value of k_0,
(3) rapidity of convergence of (10) by various aggregates of sample means and by some values of k, and
(4) relation between k_0 and the sample size n.

For the above mentioned data,

$$Y_1 = 10537/128 = 82.3203125, \qquad Y_2 = 644/128 = 5.03125,$$
$$R = Y_1/Y_2 = 16.36180242.$$

Table 2 shows other basic values for systematic samples of size $n = 1, 2, 4, \ldots, 64$ from the above data.

Table 2
Basic values by sample size

Sample size n	L_1	L_2	Lower bound for $k = L_2/2Y_2$	Optimum $k_0 := (L_2+L_1)/2Y_2$
1	1	21	2.087	2.2
2	1.5	13	1.291	1.44
4	2.75	10	0.993	1.27
8	3.25	8	0.795	1.12
16	4.0625	6	0.596	1
32	4.25	5.53125	0.549	0.97
64	4.828125	5.234375	0.520	1

Table 3–Table 9 show the rapidity of convergence of the following formula for different values of k and d for the respective samples.

$$E(\hat{R}) := R\left[1 + \frac{1}{k}\sum_{i=1}^{d-1}\left(\frac{-1}{kY_2}\right)^i\left\{\frac{\sigma_{1i}(k)}{Y_1} - \frac{\sigma_{0,i+1}(k)}{Y_2}\right\}\right] \tag{17}$$

The bold figures are true to eight places of decimals.

Table 3
$E(\hat{R})$ in (17) for $n=1$

d	$k=2.1$	$k=2.2$	$k=2.4$
2	16.5157 9119	16.4974 0238	16.4757 4387
3	16.7559 3365	16.7166 4956	16.6636 1046
4	16.9049 5575	16.8686 2462	16.8155 0784
10	17.5336 9304	17.5036 3798	17.4492 9879
30	17.9680 5665	17.9616 9800	17.9432 9082
100	18.0325 7997	18.0286 0735	18.0284 9719
250	18.0370 0152	**18.0286 9403**	**18.0286 9403**

$E(\hat{R}) = 18.0286\ 9403$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 1.6668\ 9161$

Table 4
$E(\hat{R})$ in (17) for $n=2$

d	$k=1.3$	$k=1.44$	$k=1.6$
2	16.3125 9330	16.3216 9639	16.3293 1631
3	16.6370 2766	16.5526 0757	16.4911 5358
4	16.4410 7119	16.4909 3225	16.5002 1412
10	16.5596 4712	16.6497 4705	16.6503 1687
20	16.5992 2701	16.6953 6744	16.6941 1902
40	16.6232 2528	16.7003 1365	16.7002 6155
80	16.6535 8361	**16.7003 6231**	16.7003 6228

$E(\hat{R}) = 16.7003\ 6231$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 0.3385\ 5989$

Table 5
$E(\hat{R})$ in (17) for $n=4$

d	$k=1$	$k=1.27$	$k=1.6$
2	16.5056 5904	16.4509 9325	16.4179 9569
3	16.7105 2096	16.5889 2895	16.5101 5665
4	16.5157 5623	16.5899 4190	16.5604 1195
5	16.7363 1947	16.6235 8464	16.5909 2217
10	16.5242 8329	16.6280 8595	16.6265 6390
20	16.5365 5286	16.6287 4650	16.6287 3833
30	16.5473 8619	**16.6287 4793**	16.6287 4790

$E(\hat{R}) = 16.6287\ 4793$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 0.2669\ 4551$

Table 6
$E(\hat{R})$ in (17) for $n=8$

d	$k=1$	$k=1.12$	$k=1.3$
2	16.3047 4110	16.3163 1325	16.3280 3785
3	16.3564 8650	16.3433 9717	16.3360 0749
4	16.3282 0489	16.3356 9809	16.3370 1818
5	16.3452 5632	16.3399 8261	16.3383 3634
10	16.3383 7180	16.3388 1818	16.3388 4140
15	16.3388 6266	16.3388 3008	16.3388 3048
20	16.3388 2748	**16.3388 2983**	16.3388 2985

$E(\hat{R}) = 16.3388\ 2983$, $R = 16.3618\ 0242$,
$B(\hat{R}) = -0.0229\ 7259$

Table 7
$E(\hat{R})$ in (17) for $n=16$

d	$k=0.9$	$k=1$	$k=1.1$
2	16.3722 6463	16.3702 7659	16.3688 0566
3	16.3950 4388	16.3885 7773	16.3838 2911
4	16.3880 2277	16.3892 0726	16.3881 8273
5	16.3906 7876	16.3899 6324	16.3895 1135
8	16.3899 9188	16.3900 1519	16.3900 0643
11	16.3900 1718	**16.3900 1632**	16.3900 1562

$E(\hat{R}) = 16.3900\ 1632$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 0.0282\ 1390$

Table 8
$E(\hat{R})$ in (17) for $n=32$

d	$k=0.9$	$k=0.97$	$k=1$
2	16.4530 1027	16.4403 2108	16.4356 8055
3	16.4443 6917	16.4447 5173	16.4441 5705
4	16.4462 1855	16.4458 5480	16.4456 9700
5	16.4459 2075	16.4459 4803	16.4459 2406
7	16.4459 6509	16.4459 6614	16.4459 6541
9	16.4459 6639	**16.4459 6642**	16.4459 6640

$E(\hat{R}) = 16.4459\ 6642$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 0.0841\ 6400$

Table 9
$E(\hat{R})$ in (17) for $n=64$

d	$k=0.9$	$k=1$	$k=1.1$
2	16.3722 0375	16.3702 2727	16.3687 6490
3	16.3698 9208	16.3702 2727	16.3700 3102
4	16.3702 9829	16.3702 4101	16.3702 1306
5	16.3702 3191	16.3702 4101	16.3702 3739
6	16.3702 4246	**16.3702 4103**	16.3702 4056

$E(\hat{R}) = 16.3702\ 4103$, $R = 16.3618\ 0242$,
$B(\hat{R}) = 0.0084\ 3861$

7. Mean square error of the ratio

In a similar way, we next obtain the mean square error $M(\hat{R}) = E(\hat{R}-R)^2$ and its approximate expressions given by

$$M(\hat{R}) = \frac{R^2}{k_0^2} \sum_{i=0}^{\infty} (i+1) \left(\frac{-1}{k_0 Y_2}\right)^i \left\{ \frac{\sigma_{2i}(k_0)}{Y_1^2} - \frac{2\sigma_{1,i+1}(k_0)}{Y_1 Y_2} + \frac{\sigma_{0,i+2}(k_0)}{Y_2^2} \right\}, \quad (18)$$

approximated to the second degree by

$$M(\hat{R}) := \frac{R^2}{k_0^2}\left(\frac{\sigma_{20}}{Y_1^2} - \frac{2\sigma_{11}}{Y_1 Y_2} + \frac{\sigma_{02}}{Y_2^2}\right), \tag{19}$$

to the third degree by

$$M(\hat{R}) := \frac{R^2}{k_0^2}\left\{\frac{3k_0-2}{k_0}\left(\frac{\sigma_{20}}{Y_1^2} - \frac{2\sigma_{11}}{Y_1 Y_2} + \frac{\sigma_{02}}{Y_2^2}\right) - \frac{2}{k_0}\left(\frac{\sigma_{21}}{Y_1^2 Y_2} - \frac{2\sigma_{12}}{Y_1 Y_2^2} + \frac{\sigma_{03}}{Y_2^3}\right)\right\}, \tag{20}$$

and to the fourth degree by

$$M(\hat{R}) := \frac{R^2}{k_0^2}\left\{\frac{6k_0^2-8k_0+3}{k_0^2}\left(\frac{\sigma_{20}}{Y_1^2} - \frac{2\sigma_{11}}{Y_1 Y_2} + \frac{\sigma_{02}}{Y_2^2}\right) - \frac{2(4k_0-3)}{k_0^2}\left(\frac{\sigma_{21}}{Y_1^2 Y_2} - \frac{2\sigma_{12}}{Y_1 Y_2^2} + \frac{\sigma_{03}}{Y_2^3}\right) + \frac{3}{k_0^2}\left(\frac{\sigma_{22}}{Y_1^2 Y_2^2} - \frac{2\sigma_{13}}{Y_1 Y_2^3} + \frac{\sigma_{04}}{Y_2^4}\right)\right\}. \tag{21}$$

It is clear that, when $k_0 = 1$, (10), (13)–(15) and (18)–(21) are identical to those obtained from the conventional procedure.

8. Application of the catalyzer

The technique of introducing a catalyzer can be adopted to take expectations of the expressions having statistics in the denominator or involving (square) root of statistics such as regression coefficient, correlation coefficient, coefficient of variation, etc. For example, if $\hat{Y}$ is a positive and finite variable, and Y the expectation of $\hat{Y}$, then the expectation of $\hat{Y}^{-1}$ and the mean square error of $\hat{Y}^{-1}$ for Y^{-1} are directly derived from the preceding theory by putting the numerator equal to unity. The approximate expressions to the second degree are given by

$$E(\hat{Y}^{-1}) := Y^{-1}\left(\frac{3k_0^2-3k_0+1}{k_0^3} + \frac{1}{k_0^3}\frac{\sigma_2}{Y^2}\right),$$

$$M(\hat{Y}^{-1}) := Y^{-2}\left\{\frac{(k_0-1)^3(k_0-3)}{k_0^4} - \frac{2k_0-3}{k_0^4}\frac{\sigma_2}{Y^2}\right\},$$

where $\sigma_2 = E(\hat{Y}-Y)^2$ and k_0 is the catalyzer.

The expectation of $\hat{Y}^{1/2}$ and the mean square error of $\hat{Y}^{1/2}$ for $Y^{1/2}$ are obtained by using the catalyzer which satisfies $|\hat{Y}-kY|/kY < 1$.

It follows that

$$\hat{Y}^{1/2} = (kY)^{1/2}\left(1 + \frac{\hat{Y} - kY}{kY}\right)^{1/2}$$

$$= (kY)^{1/2}\left\{1 + \frac{1}{2}\frac{\hat{Y} - kY}{kY} - \frac{1}{8}\left(\frac{\hat{Y} - kY}{kY}\right)^2 + \cdots\right\}$$

The approximate expressions to the second degree are given by

$$E(\hat{Y}^{1/2}) := Y^{1/2}\left(\frac{3k_0^2 + 6k_0 - 1}{8k_0^{3/2}} - \frac{1}{8k_0^{3/2}}\frac{\sigma_2}{Y^2}\right),$$

$$M(\hat{Y}^{1/2}) := 2Y\left(1 - \frac{3k_0^2 + 6k_0 - 1}{8k_0^{3/2}} + \frac{1}{8k_0^{3/2}}\frac{\sigma_2}{Y^2}\right).$$

This approach is applicable to the sample standard deviation as well.

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References

Koop, J.C. (1951). A note on the bias of the ratio estimate. *Bull. Internat. Statist. Inst.* 33(II), 141–146.

Funatsu, Y. (1980). A note on Koop's procedure to obtain the bias of the ratio estimate. To appear in *Sankhyā*.

Murthy, M.N. (1962). Almost unbiased estimators based on interpenetrating sub-samples. *Sankhyā*, A, 24(3), 303–314.

Murthy, M.N. (1972). Statistical methods. Lecture notes, SIAP, Tokyo, 18–19.